

# On spectra of Koopman, groupoid and quasi-regular representations.

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## 1 Introduction.

Study of spectrum of operators of unitary group representations has a long story, remarkable achievements and numerous applications. For instance, famous Kadison-Kaplanski Conjecture which was proven for the case of amenable groups by Higson and Kasparov in [32] asserts that for a torsion free group  $G$  and an element  $m \in \mathbb{C}[G]$  of the group algebra of  $G$  the spectrum of  $\lambda_G(m)$  is connected, where  $\lambda_G$  is the left regular representation of  $G$ . The remarkable Kesten's criterion of amenability and the fundamental property (T) of Kazhdan can be formulated in terms of spectral properties of operators of the form  $\lambda_G(m)$ ,  $m \in \mathbb{C}[G]$ . The topic in discussion is related to the spectral theory of graphs and networks, random walks, theory of operator algebras, discrete potential theory, abstract harmonic analysis etc.

There are three important types of unitary representations associated to a measure class preserving action of a discrete group  $G$  on a probability space  $(X, \mu)$ : quasi-regular, Koopman and groupoid representation. The goal of this article is to show that there is a close relation between spectral properties of these three types of representations.

For  $x \in X$  quasi-regular representation  $\rho_x$  acting on  $l^2(Gx)$  is a natural generalization of the regular representation  $\lambda_G$ . Spectra of quasi-regular representations play

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important role in random walks on groups and Schreier graphs (see e.g. [37]). Quasi-regular representations naturally give rise to Hecke algebras and their representations

Koopman representation (which we denote by  $\kappa$ ) acts in  $L^2(X, \mu)$ . Some important properties of the dynamical system  $(G, X, \mu)$ , such as ergodicity and weak mixing, can be reformulated in terms of spectral properties of  $\kappa$  (see e.g. [7]).

If, in addition,  $G$  is countable then groupoid representation  $\pi$  is defined in  $L^2(\mathcal{R}, \nu)$ , where  $\mathcal{R} \subset X \times X$  is the orbit equivalence relation and  $\nu$  is a measure on  $\mathcal{R}$  which is the product of  $\mu$  and the counting measure on leaves. In [50] Vershik introduced an important class of *totally non-free* actions. For an ergodic totally non-free action of a group  $G$  the groupoid representation  $\pi$  is a factor representation (that is, the center of the von Neumann algebra generated by  $\pi(G)$  is trivial; see e.g. [49] and [17]). If in addition this action is measure-preserving then  $\pi$  is a finite type factor representation. For many important examples of groups all finite type factor representations can be obtained using the groupoid construction (see e.g. [36] and [15]).

Given a unitary representation  $U$  of a group  $G$  and an element  $m \in \mathbb{C}[G]$  (or, more generally  $m \in l^1(G)$ ) define Hecke type operators

$$U(m) = \sum_{s \in G} m(s)U(s).$$

For an operator  $A$  denote by  $\sigma(A)$  its spectrum. The main result of the present paper is the following:

**Theorem 1.** 1) *For an ergodic measure class preserving action of a countable group  $G$  on a standard probability space  $(X, \mu)$  and any  $m \in \mathbb{C}[G]$  one has*

$$\sigma(\kappa(m)) \supset \sigma(\rho_x(m)) = \sigma(\pi(m)) \quad (1)$$

*for  $\mu$ -almost all  $x \in X$ , where  $\kappa$  is the Koopman representation,  $\pi$  is the groupoid representation associated to the action of  $G$  on  $X$ ,  $\rho_x$  are the corresponding quasi-regular representations.*

2) *If, moreover,  $(G, X, \mu)$  is hyperfinite, then one has*

$$\sigma(\kappa(m)) = \sigma(\pi(m)). \quad (2)$$

3) *If, in addition to the conditions of 1),  $\mu$  is  $G$ -invariant and non-atomic, then  $\sigma(\kappa_0(m)) = \sigma(\pi(m))$ , where  $\kappa_0$  is the restriction of  $\kappa$  onto the orthogonal complement of constant functions in  $L^2(X, \mu)$ .*

This result is closely related to the topic of weak containment of representations. Given a unitary representation  $\rho$  let  $C_\rho$  denote the  $C^*$ -algebra generated by operators  $\rho(g), g \in G$ . Recall that a unitary representation  $\rho$  of a group  $G$  is weakly contained

into  $\eta$  (denoted by  $\rho \prec \eta$ ) if there exists a surjective homomorphism  $\phi : C_\eta \rightarrow C_\rho$  such that  $\phi(\eta(g)) = \rho(g)$  for all  $g \in G$  (see also Definition 5). We will write  $\rho \sim \eta$  if  $\rho$  is weakly equivalent to  $\eta$  (i.e.  $\rho \prec \eta$  and  $\eta \prec \rho$ ). An action  $(G, X, \mu)$  is called hyperfinite if the (countable) orbit equivalence relation associated to this action is hyperfinite with respect to  $\mu$  (see e.g. [16]). Theorem 1 can be rewritten in terms of weak containment of representations. Namely, (1) means that

$$\kappa \succ \rho_x \sim \pi$$

for  $\mu$ -almost all  $x \in X$  and (2) means that  $\kappa \sim \pi$ .

As example of an application of relations between spectra of representations is related to the torsion group  $\mathcal{G} = \langle a, b, c, d \rangle$  of intermediate growth constructed by the second author in [21] and studied in [22] and other papers. Recall that  $\mathcal{G}$  acts naturally on the boundary  $\partial T$  of a binary rooted tree  $T$  (see e.g. [25]). We show the following:

**Theorem 2.** *The spectrum of the Cayley graph of  $\mathcal{G}$  (i.e. the spectrum of  $\lambda_{\mathcal{G}}(a + b + c + d)$ ) is  $[-2, 0] \cup [2, 4]$  and coincides with the spectrum of the Schreier graph  $\Gamma_x$  of the action of  $\mathcal{G}$  on  $\partial T$  for any  $x \in \partial T$ .*

In fact, our proof shows that the spectra of  $\lambda_{\mathcal{G}}(ta + b + c + d)$  and  $\rho_x(ta + b + c + d)$  coincide for any  $t \in \mathbb{R}$  and are equal to a union of two intervals, an interval, or two points. In [26] the authors studied the operator  $\rho_x(ta + ub + vc + wd)$  for parameters  $t, u, v, w \in \mathbb{R}$  such that  $t \neq 0$ ,  $u \neq -v$ ,  $u \neq -w$ ,  $v \neq -w$  and at least two of  $v, w, t$  are distinct. They showed that the spectrum of  $\rho_x(ta + ub + vc + wd)$  is a Cantor set of Lebesgue measure zero by reduction to the results known for random Schrödinger operator and substitutional dynamical system. The substitution  $\tau$  involved

$$\tau : a \rightarrow aca, b \rightarrow d, c \rightarrow b, d \rightarrow c$$

appears in the presentation

$$\mathcal{G} = \langle a, b, c, d | a^2, b^2, c^2, d^2, bcd, \tau^i((ad)^4), \tau^i((adacac)^4) \rangle \quad (\text{see [35]}).$$

An interesting question is whether spectra of  $\lambda_{\mathcal{G}}(ta + ub + vc + wd)$  and  $\rho_x(ta + ub + vc + wd)$  coincide for arbitrary parameters  $t, u, v, w \in \mathbb{R}$ .

Notice that there are not many examples of groups for which the spectrum of the Cayley graph is calculated. Theorem 1 is the first case when the spectrum of the Cayley graph is computed for a group of intermediate growth. The coincidence of the spectra of Schreier graphs  $\Gamma_x$  and the Cayley graph of  $\mathcal{G}$  is very surprising since  $\Gamma_x$  are of linear growth and are very different from the Cayley graph of  $\mathcal{G}$ .

Observe that  $\mathcal{G}$  has an abelian extension  $\tilde{\mathcal{G}}$  which is a torsion free group of intermediate growth generated by four elements  $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$  (see [22]). From amenability of  $\tilde{\mathcal{G}}$

and Proposition 3.7 from [4] one can deduce that the spectrum of the Cayley graph of  $\tilde{\mathcal{G}}$  is  $[-4, 4]$ .

Theorem 1 has a few applications. Among them is an application to branch and weakly branch groups. A group acting on a rooted tree  $T$  is called weakly branch if it acts transitively on each level of the tree and for every vertex  $v$  of  $T$  it has a nontrivial element  $g$  supported on the subtree  $T_v$  emerging from  $v$  (see e.g. [5] and [25]). Branch groups play important role in many investigations in group theory and around (see [24], [5] and [25]). The class of branch groups contains groups of intermediate growth, amenable but not elementary amenable groups, groups with finite commutator width etc. Weakly branch groups is a natural generalization of the class of branch groups playing important role in holomorphic dynamics (see [42]) and in the theory of fractals (see [28]). In Section 4 using Theorem 1 and results of [14] and [4] we will show that any subexponentially bounded weakly branch group  $G$  admits uncountably many pairwise disjoint (not unitary equivalent) pairwise weakly equivalent irreducible representations.

We will finish the paper by showing two examples of computation of spectra of Hecke type operators of representations of the group  $\mathcal{G}$  associated to the action of  $\mathcal{G}$  on the boundary of a binary rooted tree. These examples will illustrate the method of operator recursions (used in [4] and other places).

## 2 Preliminaries.

In this section we give necessary preliminaries on representation theory and related topics. We will mostly deal with actions of countable groups on a standard probability space (or Lebesgue space). A probability space is standard, if it is isomorphic modulo zero measure to an interval with Lebesgue measure, a finite or countable set of atoms, or a combination (disjoint union) of both. We refer the reader to [45] or [20] for details.

### 2.1 Koopman and quasi-regular representations.

The most natural representations that one can associate to a measure-preserving action of a group  $G$  on a measure space  $(X, \mu)$ , where  $\mu$  is a quasi-invariant probability measure, is the Koopman representation  $\kappa$  of  $G$  in  $L^2(X, \mu)$  acting by:

$$(\kappa(g)f)(x) = \sqrt{\frac{d\mu(g^{-1}(x))}{d\mu(x)}} f(g^{-1}x),$$

where  $\sqrt{\frac{d\mu(g^{-1}(x))}{d\mu(x)}}$  is the Radon-Nikodym derivative. This representation is important due to the fact that the spectral properties of  $\kappa$  reflect the dynamical properties of the action such as ergodicity and weak-mixing. It is known that for an ergodic action

operators  $\kappa(g)$  together with operators of multiplication by functions from  $L^\infty(X, \mu)$  generate the algebra of all bounded operators on  $L^2(X, \mu)$ . One of the most natural questions concerning Koopman representations is whether it is irreducible. There are several examples of group actions with quasi-invariant measures known for which  $\kappa$  is irreducible (see e.g. [3], [6], [11], [18], [19] and [39]), but the general case is still not well understood. In [14] we constructed a new class of examples of irreducible Koopman representations arising from subexponential actions of weakly branch groups on boundaries of rooted trees. We will talk about this construction in more details in Section 4.

Given a countable group acting on a set  $X$  and a point  $x \in X$  one can define the quasi-regular representation  $\rho_x$  in  $l^2(Gx)$ , where  $Gx$  is the orbit of  $x$ , by:

$$(\rho_x(g)f)(y) = f(g^{-1}y).$$

Notice that the isomorphism class of  $\rho_x$  depends only on the stabilizer  $\text{St}_G(x)$  of  $x$ .

The question of irreducibility and disjointness of quasi-regular representations was studied by Mackey. Recall that two subgroups  $H_1, H_2$  of a group  $G$  are called *commensurable* if  $H_1 \cap H_2$  is of finite index in both  $H_1$  and  $H_2$ . The groups  $H_1$  and  $H_2$  are called *quasi-conjugate* in  $G$  if  $gH_1g^{-1}$  is commensurable to  $H_2$  for some  $g \in G$ . By definition, *commensurator* of  $H < G$  is the subgroup of  $G$  defined by

$$\text{comm}_G(H) = \{g \in G : H \cap gHg^{-1} \text{ has finite index in } H \text{ and } gHg^{-1}\}.$$

Mackey showed the following (see [40], Corollary 7):

**Theorem 3** (Mackey). 1) *Let  $H$  be a subgroup of an infinite discrete countable group  $G$ . Then the quasi-regular representation  $\rho_{G/H}$  is irreducible if and only if  $\text{comm}_G(H) = H$ .* 2) *Let  $H_1, H_2$  be two subgroups of  $G$  such that  $\text{comm}_G(H_i) = H_i$ ,  $i = 1, 2$ . Then  $\rho_{G/H_1}$  and  $\rho_{G/H_2}$  are unitary equivalent if and only if  $H_1$  and  $H_2$  are quasi-conjugate.*

In [4] the authors proved the following:

**Proposition 1.** *Let  $G$  be a weakly branch group,  $T$  be the corresponding spherically homogeneous rooted tree,  $x \in \partial T$  and  $\text{St}_G(x)$  be its stabilizer. Then  $\text{comm}_G(\text{St}_G(x)) = \text{St}_G(x)$ .*

Theorem 3 together with Proposition 1 immediately imply:

**Corollary 1.** *For a weakly branch group  $G$  and any  $x \in \partial T$  the quasi-regular representation  $\rho_x$  is irreducible.*

In addition, in [14] the authors of the present paper showed using Theorem 3 that the representations  $\rho_x$  associated to an action of a weakly branch group on the boundary of a rooted tree for  $x \in \partial T$  from different orbits are pairwise disjoint.

## 2.2 Groupoid construction and factor representations.

Here we briefly recall some important notions and facts from Theory of Operator Algebras (see [8], [33] and [34] for details). An algebra  $\mathcal{M}$  of operators acting in a Hilbert space is called a von Neumann (or shortly  $W^*$ -) algebra if it is closed in the weak operator topology. A  $W^*$ -algebra  $\mathcal{M}$  is called a factor if its center is trivial. Notice that by disintegration theorem (see [48], Theorem 8.21), each von Neumann algebra can be written as a direct integral of factors. In this paper we will be concerned only with finite type  $W^*$ -algebras, that is those which admit a faithful finite trace (normal, normalized, positive-definite function on the algebra).

A unitary representation  $\pi$  of a group  $G$  is called *factor representation* if the  $W^*$ -algebra  $\mathcal{M}_\pi$  generated by operators  $\pi(g), g \in G$ , is a factor. Two unitary representations  $\pi_1, \pi_2$  of  $G$  are called *quasi-equivalent* if there is a von Neumann algebra isomorphism  $\omega : \mathcal{M}_{\pi_1} \rightarrow \mathcal{M}_{\pi_2}$  such that

$$\omega(\pi_1(g)) = \pi_2(g) \text{ for all } g \in G.$$

Let  $(X, \mu)$  be a standard probability space with a measure class preserving action of a countable group  $G$  on it. Denote by  $\mathcal{R}$  the orbit equivalence relation on  $X$ . For  $A \subset X^2$  and  $x \in X$  set  $A_x = A \cap (X \times \{x\})$ ,  $A^x = A \cap (\{x\} \times X)$ . Introduce measures  $\nu_l, \nu_r$  on  $\mathcal{R} \subset X^2$  by

$$\nu_l(A) = \int_X |A^x| d\mu(x), \quad \nu_r(A) = \int_X |A_x| d\mu(x).$$

Notice that if  $\mu$  is invariant with respect to  $G$  then  $\nu_l = \nu_r$ . If  $\mu$  is only quasi-invariant with respect to  $G$  then one can define a Radon-Nikodim derivative  $D(x, y) = \frac{d\nu_l}{d\nu_r}(x, y)$ . In fact, one has (see [16]):

$$\frac{d\mu(gx)}{d\mu(x)} = D(x, gx) \text{ for all } g \in G \text{ and almost all } x \in X.$$

**Definition 1.** The (left) groupoid representation of  $G$  is the unitary representation  $\pi$  in  $L^2(\mathcal{R}, \nu_r)$  defined by

$$(\pi(g)f)((x, y)) = f(g^{-1}x, y).$$

Let  $\xi$  the unit vector given by  $\xi(x, y) = \delta_{x,y} \in L^2(\mathcal{R}, \nu)$ , where  $\delta_{x,y}$  is the Kronecker delta-symbol. For a measurable automorphism  $g$  of  $X$  denote by  $\text{Fix}(g)$  the set of fixed points of  $g$ :  $\text{Fix}(g) = \{x \in X : gx = x\}$ . Observe that

$$(\pi(g)\xi, \xi) = \mu(\text{Fix}(g)) \text{ for all } g \in G.$$

Namely, for a function  $m \in L^\infty(X, \mu)$  introduce operators  $m_l : L^2(\mathcal{R}, \nu_r) \rightarrow L^2(\mathcal{R}, \nu_r)$  by

$$(m_l f)(x, y) = m(x)f(x, y).$$

Let  $\mathcal{M}_{\mathcal{R}}$  stand for the  $W^*$ -algebra generated by  $\mathcal{M}_{\pi}$  and operators  $m_l, m \in L^2(X, \mu)$ . This algebra is sometimes referred to as Murray-von Neumann or Krieger algebra. Observe that for an ergodic action of a group  $G$  the algebra  $\mathcal{M}_{\mathcal{R}}$  is a factor (see [49], Theorem 2.10, or [17], Proposition 2.9(2)) and vector  $\xi$  is cyclic and separating for  $M_{\mathcal{R}}$  (see [17], Proposition 2.5).

For us it will be important that the groupoid representations can be written as direct integrals of quasi-regular ones.

**Proposition 2.** *The groupoid representation  $\pi$  is unitary equivalent to the representation  $\int_X \rho_x d\mu(x)$ .*

The above statement seems folklore and is mentioned, for example, in [49], §2.

Similarly to representation  $\pi$  of  $G$  in  $L^2(\mathcal{R}, \nu_r)$  one can introduce a representation  $\tilde{\pi}$  of  $G$  in  $L^2(\mathcal{R}, \nu_l)$  by

$$(\tilde{\pi}(g)f)(x, y) = \sqrt{\frac{d\mu(g^{-1}x)}{d\mu(x)}} f(g^{-1}x, y). \quad (3)$$

It is straightforward to verify that the representation  $\tilde{\pi}$  is unitary equivalent to  $\pi$  via the intertwining isometry  $\mathcal{I} : L^2(\mathcal{R}, \nu_r) \rightarrow L^2(\mathcal{R}, \nu_l)$  given by:

$$(\mathcal{I}f)(x, y) = \frac{1}{\sqrt{D(x, y)}} f(x, y).$$

The latter is well-defined since  $D(x, y) \neq 0$  for  $\nu_r$ -almost all  $(x, y) \in \mathcal{R}$ .

### 2.3 Hecke type operators.

Let  $U$  be a unitary representation of a countable group  $G$  and  $m \in \mathbb{C}[G]$ , that is  $m : G \rightarrow \mathbb{C}$  is a function of finite support. One can associate to  $m$  a Hecke type operator

$$U(m) = \sum_{s \in G} m(s) U(s).$$

Additionally, given  $\nu \in l^1(G)$  one can associate to it a Hecke type operator

$$U(\nu) = \sum_{s \in G} \nu(s) U(s).$$

An interesting particular case is when  $\nu$  is a measure on  $G$  (i.e.  $\nu \in l^1(G)$  with  $\nu(s) \geq 0$  for all  $s \in G$ ).

For the case of quasi-regular representations spectral properties of this operators are related to properties of random walks on graphs. In this case a fruitful study of these operators was done by Kesten. He showed that for any discrete group  $G$ ,

any symmetric generating (support generates  $G$ ) probability measure  $\nu$  on  $G$  and any normal subgroup  $H$  of  $G$  one has  $\|\rho_{G/H}(\nu)\| = \|\lambda_G(\nu)\|$  if and only if  $H$  is amenable, where  $\rho_{G/H}$  is the quasi-regular representation associated to the action of  $G$  on  $G/H$  and  $\lambda_G$  is the regular representation of  $G$ .

**Definition 2.** Let  $\rho$  and  $\eta$  be two unitary representations of a group  $G$  acting in Hilbert spaces  $\mathcal{H}_\rho$  and  $\mathcal{H}_\eta$  correspondingly. Then  $\rho$  is weakly contained in  $\eta$  (denoted by  $\rho \prec \eta$ ) if for any  $\epsilon > 0$ , any finite subset  $S \subset G$  and any vector  $v \in \mathcal{H}_\rho$  there exists a finite collection of vectors  $w_1, \dots, w_n \in \mathcal{H}_\eta$  such that

$$|(\rho(g)v, v) - \sum_{i=1}^n (\eta(g)w_i, w_i)| < \epsilon$$

for all  $g \in S$ .

In the language of  $C^*$ -algebras weak inclusion  $\rho \prec \eta$  means that there exists a surjective homomorphism  $\phi : C_\eta \rightarrow C_\rho$  such that  $\phi(\eta(g)) = \rho(g)$  for all  $g \in G$ . It is not hard to see that

$$\text{if } \rho \prec \eta \text{ then } \sigma(\rho(\nu)) \subset \sigma(\eta(\nu)) \text{ for all } \nu \in l^1(G). \quad (4)$$

In fact, in [12] Dixmier showed the following:

**Theorem 4.** Let  $G$  be a discrete group and  $\rho, \eta$  be two unitary representations of  $G$ . Then  $\rho \prec \eta$  if and only if  $\|\rho(\nu)\| \leq \|\eta(\nu)\|$  for every  $\nu \in l^1(G)$ .

**Remark 1.** Let  $\rho, \eta$  be two unitary representations of a discrete group  $G$  and  $\sigma(\rho(m)) \subset \sigma(\eta(m))$  for all  $m \in \mathbb{C}[G]$ . Then  $\|\rho(m)\| \leq \|\eta(m)\|$  for every  $m \in \mathbb{C}[G]$  and by continuity  $\|\rho(\nu)\| \leq \|\eta(\nu)\|$  for every  $\nu \in l^1(G)$ . Hence  $\rho \prec \eta$ .

Using Theorem 4, (4), Remark 1 and the fact that  $\|A\| = \|A * A\|^{\frac{1}{2}}$  for any bounded linear operator  $A$  on a Hilbert space we obtain:

**Corollary 2.** Let  $\rho, \eta$  be two unitary representations of a discrete group  $G$ . Then the following conditions are equivalent:

- 1)  $\rho \prec \eta$ ;
- 2)  $\sigma(\rho(\nu)) \subset \sigma(\eta(\nu))$  for all  $\nu \in l^1(G)$ ;
- 3)  $\|\rho(m)\| \leq \|\eta(m)\|$  for every positive  $m \in \mathbb{C}[G]$ .
- 4) there exists a surjective homomorphism  $\phi : C_\eta \rightarrow C_\rho$  such that  $\phi(\eta(g)) = \rho(g)$  for all  $g \in G$ .



Here positiveness of an element  $m$  of some  $C^*$ -algebra  $A$  means that it can be represented as  $m = x^*x$  with  $x \in A$ . Equivalently,  $m$  is positive if  $m$  is self-adjoint ( $m = m^*$ ) and  $\sigma(m) \subset \mathbb{R}_+ = [0, \infty)$ . Recall that for a unit positive element  $m$  one has  $1 \in \sigma(m)$ .

For an action of a group  $G$  on a measure space  $(X, \mu)$  denote by  $\mathcal{R} = \mathcal{R}_G$  the equivalence relation generated by  $G$  on  $X$ . In 1977 Zimmer introduced a notion of amenability of an ergodic action of  $G$  on a measure space  $(X, \mu)$  with a quasi-invariant probability measure  $\mu$ . Later Adams, Elliott and Giorgano [1] showed that for any separable locally compact group  $G$  Zimmer's amenability is equivalent to the following two conditions:

1)  $\mathcal{R}_G$  is  $\mu$ -hyperfinite (i.e. on a set of full measure it is equal to a union of finite measurable equivalence relations);

2) for  $\mu$ -almost all  $x \in X$  the stabilizer  $\text{St}_G(x)$  is amenable.

Observe that condition 1) is equivalent to the following (see e.g. [16], Proposition 4.1):

1') on a set of full measure  $\mathcal{R}_G$  coincides with  $\mathcal{R}_{\mathbb{Z}}$  for some action of  $\mathbb{Z}$  on  $(X, \mu)$ .

**Theorem 5** (Kuhn). *For an ergodic Zimmer amenable measure class preserving action of  $G$  on a probability measure space  $(X, \mu)$  one has*

$$\kappa \prec \lambda_G,$$

where  $\kappa$  is the Koopman representation associated to the action of  $G$  and  $\lambda_G$  is the regular representation.

At the end of Section 3.3 we will derive Kuhn's Theorem from part 2) of Theorem 1. We refer the reader to [2] for a generalization of Kuhn's Theorem for locally compact groups  $G$  and a detailed discussion of a relation between amenability of an action and weak containment of the corresponding Koopman representation into the regular representation.

Another result closely related to the present paper is the following (see [43], Theorem 30):

**Theorem 6** (Pichot). *A measure class preserving action of a countable group  $G$  on a standard probability space  $L^2(X, \mu)$  is hyperfinite if and only if for every  $m \in l^1(G)$  with  $\|m\|_1 = 1$  one has  $\|\pi(m)\| = 1$ , where  $\pi$  is the corresponding groupoid representation.*

Observe that the original result of Pichot concerns arbitrary equivalence relations on  $(X, \mu)$  (not necessarily generated by group actions). Theorem 6 is the particular case which is of interest to us. Theorem 1 implies the "if" direction of Theorem 6.

The following Proposition is a common lore.

**Proposition 3.** *Let  $H < G$  be any subgroup. Then  $\rho_{G/H} \prec \lambda_G$  if and only if  $H$  is amenable.*

Observe that the above statement is not true for locally compact groups. Namely, in [2] (see Section 4.2) Anantaraman-Delaroche presented an example (accredited to Bekka) of a locally compact group  $G$  with a closed non-amenable subgroup  $H$  such that  $\rho_{G/H} \prec \lambda_G$ .

The following result was the starting point of our investigation:

**Theorem 7** (Bartholdi - Grigorchuk). *Let  $G$  be a finitely generated group acting on a regular rooted tree  $T$  and  $m \in \mathbb{C}[G]$ . Then  $\sigma(\rho_x(m)) \subset \sigma(\kappa(m))$  for all  $x \in \partial T$ . If moreover the Schreier graph of the action of  $G$  on the orbit  $Gx$  of  $x \in X$  is amenable, then  $\sigma(\rho_x(m)) = \sigma(\kappa(m))$ .*

For the proof of Theorem 7 we refer the reader to [25], Proposition 10.4 (see also [4], Theorem 3.6).

In [4] Bartholdi and the second author introduced a new method for calculating spectra of Hecke type operators associated to actions of self-similar groups. The method is based on using operator recursions (relying on self-similar properties of a group) and Schur complement trick (in finite or infinite-dimensional cases). Let  $\mathcal{G}$  be the 3-generated torsion 2-group of intermediate growth constructed by the second author in [21]. This group acts on a binary rooted tree  $T$ . Let  $\partial T$  be its boundary supplied by the uniform Bernoulli measure  $\mu = \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{N}}$ . Let  $S = \{a, b, c, d\}$  be the set of generators of  $\mathcal{G}$  and

$$\Delta = \frac{1}{|S|} \sum_{s \in S} s \in \mathbb{C}[\mathcal{G}].$$

One of the results of [4] is that for all  $x \in \partial T$  one has

$$\sigma(\rho_x(\Delta)) = \sigma(\kappa(\Delta)) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1].$$

Another result is construction of examples of groups for which  $\sigma(\kappa(\Delta))$  is a Cantor set which can be calculated explicitly. For the survey on developing the method and using it for calculations of spectra of Shreier graphs and associated with them fractals see [28].

### 3 Proof of Theorem 1.

For convenience, we will split Theorem 1 into three parts: Propositions 4, 8 and 9.

#### 3.1 Equivalence of quasi-regular and groupoid representations.

**Proposition 4.** *For an ergodic measure class preserving action of a countable group  $G$  on a standard probability space  $(X, \mu)$  one has  $\rho_x \sim \pi$  for almost all  $x \in X$ .*

The proof is based on a few technical statements. We will formulate these statements, deliver Proposition 4 from them and then give the proofs of the statements.

For an action of a group  $G$  on a space  $X$ , a collection  $S = \{g_1, g_2, \dots, g_n\} \in G, n \in \mathbb{N}$  and a point  $x \in X$  introduce an *orbital graph*  $\Gamma_x = \Gamma_{x, g_1, \dots, g_n}$  as a marked rooted graph whose vertex set is the set of points of the orbit  $Gx$  ( $x$  is the root) and such that  $y, z \in Gx$  are connected by a directed edge marked by  $g_i$  if and only if  $z = g_i y$ . Notice that here we don't assume that the group  $G$  is generated by  $S$  and so the graphs  $\Gamma_x$  are not necessary connected. In case if  $S$  generates  $G$  orbital graphs coincide with marked Schreier graphs corresponding to  $S$ . Fix a numeration of all elements of  $G$ :

$$G = \{s_1, s_2, s_3, \dots\} \text{ with } s_1 = e, \text{ the unit element of } G. \quad (5)$$

For  $k \in \mathbb{N}$ ,  $x \in X$  and  $y \in Gx$  denote by  $B_k(y)$  the subgraph of  $\Gamma_x$  consisting of vertices  $\{z \in Gx : z = s_i y \text{ for some } i \leq k\}$  and all edges connecting them. We denote  $y$  the root of  $B_k(y)$ . Observe that  $B_k(y)$  may be disconnected and that

$$\Gamma_x = \bigcup_{k \in \mathbb{N}} B_k(x).$$

**Definition 3.** We will say that two orbital graphs  $\Gamma_x$  and  $\Gamma_y$  are locally isomorphic if for any  $k$  there exist a vertex  $u$  of  $\Gamma_x$  and a vertex  $v$  of  $\Gamma_y$  such that  $B_k(u)$  is isomorphic (as marked rooted graph) to  $B_k(y)$  and  $B_k(v)$  is isomorphic to  $B_k(x)$ .

The following statement is a straightforward modification of Proposition 8.11 from [25].

**Proposition 5.** *Let  $G$  act ergodically by measure class preserving transformations on a standard probability space  $(X, \mu)$ . Then there exists a subset  $A \subset X$  of a full measure such that for any  $g_1, \dots, g_n \in G, n \in \mathbb{N}$  and any  $x, y \in A$  the marked orbital graphs  $\Gamma_{x, g_1, \dots, g_n}$  and  $\Gamma_{y, g_1, \dots, g_n}$  are locally isomorphic.*

For  $m \in \mathbb{C}[G]$  denote the support of  $m$  by

$$\text{supp}(m) = \{g \in G : m(g) \neq 0\}.$$

**Proposition 6.** *Let  $G$  act on a space  $X$ ,  $x, y \in X$  and  $m \in \mathbb{C}[G]$ . Let  $\text{supp}(m) = \{g_1, g_2, \dots, g_n\}$ . If the orbital graphs  $\Gamma_{x, g_1, \dots, g_n}$  and  $\Gamma_{y, g_1, \dots, g_n}$  are locally isomorphic then  $\sigma(\rho_x(m)) = \sigma(\rho_y(m))$ .*

The proof of Proposition 6 will be provided below.

The next Proposition is a standard statement about direct integral of Hilbert spaces. It can be derived from Lemma 2, [10]. It is easy to see that all conditions of Lemma 2, [10], are satisfied in our case.

**Proposition 7.** *Let  $(X, \mu)$  be a standard probability space,  $\mathcal{H} = \int_X \mathcal{H}_x d\mu(x)$  be a direct integral of separable Hilbert spaces,  $M_x$  be an integrable family of operators and  $M = \int M_x d\mu(x)$ . If the spectrum  $\sigma(M_x)$  of almost all operators  $M_x$  coincide and is equal to  $\sigma$  then  $\sigma(M) = \sigma$ .*

Now, let us derive Proposition 4 from the above statements.

**Proof of Proposition 4.** Let  $m \in \mathbb{C}[G]$  and  $\text{supp}(m) = \{g_1, \dots, g_n\}$ . By Proposition 5 for almost all  $x$  the orbital graphs  $\Gamma_{x, g_1, \dots, g_n}$  are pairwise locally isomorphic. Proposition 6 implies that the spectra  $\sigma(\rho_x(m))$  coincide for almost all  $x$ . Denote this spectrum by  $\sigma$ . From Proposition 7 we get that the spectrum of

$$\int_X \rho_x(m) d\mu(x)$$

is equal to  $\sigma$ . From Proposition 2 we get that  $\sigma(\pi(m)) = \sigma$ . Finally, Corollary 2 implies that  $\pi \sim \rho_x$  for almost all  $x \in X$ .

**Proof of Proposition 5.** Fix  $n$  and  $S = \{g_1, g_2, \dots, g_n\} \in G$ . Let us call a finite rooted directed graph with edges marked by elements of  $S$   $r$ -admissible if it is isomorphic to  $B_r(x)$  (as marked rooted graph) for some point  $x \in X$ . For an  $r$ -admissible graph  $\Delta$  denote by  $X_\Delta(r)$  the set of points  $y \in X$  such that  $B_r(y)$  is isomorphic to  $\Delta$ . For any fixed  $r$  the sets  $X_\Delta(r)$ , where  $\Delta$  is  $r$ -admissible, cover  $X$ , therefore, there exist  $\Delta$  for which  $X_\Delta(r)$  is of positive measure. We will call such  $\Delta$  positively  $r$ -admissible. Let  $P_r$  be the set of positively  $r$ -admissible graphs and  $Z_r$  be the set of  $r$ -admissible but not positively  $r$ -admissible graphs. For an  $r$ -admissible graph  $\Delta$  set

$$\tilde{X}_\Delta(r) = \bigcup_{g \in G} g(X_\Delta(r)).$$

Clearly, for  $\Delta \in P_r$  the set  $\tilde{X}_\Delta(r)$  is an invariant set of positive measure. Since the action is ergodic  $\mu(\tilde{X}_\Delta(r)) = 1$ . For  $\Delta \in Z_r$  one has  $\mu(X_\Delta(r)) = 0$ . Denote

$$X_*^S = \bigcap_{r \geq 1} \bigcap_{\Delta \in P_r} \tilde{X}_\Delta(r) \setminus \left( \bigcup_{r \geq 1} \bigcup_{\Delta \in Z_r} X_\Delta(r) \right).$$

Then  $\mu(X_*^S) = 1$ .

Further, let  $x, y \in X_*^S$ ,  $r \in \mathbb{N}$  and  $\Delta = B_r(x)$ . Definition of  $X_*^S$  implies that  $\Delta \in P_r$ . Therefore,  $y \in \tilde{X}_\Delta(r)$ . Thus,  $y \in g(X_\Delta(r))$  for some  $g \in G$ . This means that  $B_r(g^{-1}y) = \Delta = B_r(x)$ . We obtain that for any  $x, y \in X_*^S$  the orbital graphs  $\Gamma_{x, g_1, \dots, g_n}$  and  $\Gamma_{y, g_1, \dots, g_n}$  are locally isomorphic.

Finally, denoting by  $A$  the intersection of all sets of the form  $X_*^S$  where  $S$  is a finite subset of  $G$  we obtain the desired.

**Proof of Proposition 6.** Let  $G, X, x, y, m$  be as in the formulation of the Proposition 6 and the orbital graphs  $\Gamma_x = \Gamma_{x, g_1, \dots, g_n}$  and  $\Gamma_y = \Gamma_{y, g_1, \dots, g_n}$  be locally isomorphic. Set

$$R = 2 \sum_{g \in \text{supp}(m)} |m(g)|.$$

Fix a point  $\alpha$  from  $\sigma(\rho_x(m))$  and let us show that  $\alpha \in \sigma(\rho_y(m))$ . Clearly,  $\alpha \leq \frac{1}{2}R$ . The proof of the following Lemma is straightforward and we omit it here.

**Lemma 1.** *Let  $A$  be any bounded nonzero linear operator in a Hilbert space and  $R \geq 2\|A\|$ . Then*

$$\alpha \in \sigma(A) \Leftrightarrow 1 \in \sigma(I - \frac{1}{R^2}(A - \alpha I)(A - \alpha I)^*),$$

where  $I$  is the identity operator.

Using Lemma 1 we obtain that for any unitary representation  $\omega$  one has:

$$\alpha \in \sigma(\omega(m)) \Leftrightarrow 1 \in \sigma(I - \frac{1}{R^2}(\omega(m) - \alpha I)(\omega(m) - \alpha I)^*).$$

The operator  $I - \frac{1}{R^2}(\omega(m) - \alpha I)(\omega(m) - \alpha I)^*$  is of the form  $\omega(s)$  for some  $s \in \mathbb{C}[G]$ , positive and of norm less or equal to 1. It follows that without loss of generality we may assume that  $\alpha = 1$  and operators  $\rho_x(m)$  and  $\rho_y(m)$  are positive of norm less or equal to 1 (in fact,  $\|\rho_x(m)\| = 1$ , since we assume that  $\alpha = 1 \in \sigma(\rho_x(m))$ ).

Further, consider orbital graphs  $\Gamma_x$  and  $\Gamma_y$ . Let  $\epsilon > 0$ . Since

$$\sup_{\xi: \|\xi\|=1} (\rho_x(m)\xi, \xi) = 1$$

we can find  $l \in \mathbb{N}$  and a vector  $\eta \in l^2(G(x))$  supported on  $B_l(x)$  such that  $(\rho_x(m)\eta, \eta) > 1 - \epsilon$ . Let  $v \in \Gamma_y$  be such that  $B_l(v) \subset \Gamma_y$  is isomorphic (as a rooted labeled graph) to  $B_l(x)$ . Let  $\eta' \in l^2(B_l(v)) \subset l^2(\Gamma_y)$  be a copy of  $\eta$  via this isomorphism. Since on  $B_l(x)$  and  $B_l(v)$  the actions of elements from  $\text{supp}(m)$  respect this isomorphism, we get:

$$(\rho_x(m)\eta', \eta') = (\rho_y(m)\eta, \eta) > 1 - \epsilon.$$

It follows that  $\|\rho_y(m)\| = 1$  and  $1 \in \sigma(\rho_y(m))$ . This finishes the proof of Proposition 6.

### 3.2 Weak containment of quasi-regular representations into Koopman representation.

**Proposition 8.** 1) *Let  $G$  act on a standard probability space  $(X, \mu)$ , where  $\mu$  is a quasi-invariant measure. Let  $\kappa$  be the corresponding Koopman representation in  $L^2(X, \mu)$  and  $m \in \mathbb{C}[G]$ . Then for almost all  $x \in X$  one has  $\rho_x \prec \kappa$ .*

2) If moreover  $\mu$  is  $G$ -invariant and non-atomic then for almost all  $x \in X$  one has  $\rho_x \prec \kappa_0$ , where  $\kappa_0$  is the restriction of  $\kappa$  onto the orthogonal complement to constant functions.

One of the ingredients of the proof is the following statement:

**Lemma 2.** *Let  $T$  be a measure class preserving transformation of a standard probability space  $(X, \mu)$  such that  $Tx \neq x$  for almost all  $x \in A$ , where  $\mu(A) > 0$ . Then there exists  $B \subset A$ ,  $\mu(B) > 0$  such that  $\mu(B \cap TB) = 0$ .*

For the case of a measure preserving automorphism Lemma 2 follows from the proposition of §1 of [46]. In fact, the same proof works in the case of a measure class preserving transformation. For the reader's convenience we provide here the arguments from [46], §1.

*Proof.* Let us show first that there exists a measurable subset  $C \subset A$  such that  $\mu(T(C)\Delta C) \neq 0$ . Fix a basis  $\{A_i\}$  in  $A$ . Set

$$B_i = (A \setminus A_i) \cap T(A_i) \cup A_i \cap (A \setminus T(A_i)).$$

By definition of basis for almost all  $x, y \in A$  such that  $x \neq y$  there exists  $A_i$  such that either  $x \in A_i, y \in A \setminus A_i$  or  $y \in A_i, x \in A \setminus A_i$ . It follows that for almost all  $x \in A$  there exists  $i$  such that  $x \in B_i$ . Therefore,  $\mu(\cup B_i) = \mu(A) > 0$  and  $\mu(B_i) > 0$  for some  $i$ . Set  $C = B_i$ .

Now, if  $\mu(C \setminus TC) \neq 0$  we can set  $B = C \setminus TC$ . If  $\mu(TC \setminus C) \neq 0$  we can set  $B = T^{-1}(TC \setminus C)$ .  $\square$

**Lemma 3.** *Let  $G$  act on  $(X, \mu)$ , where  $\mu$  is a quasi-invariant probability measure. Let  $g_1, g_2, \dots, g_n \in G$ . For  $x \in X$  and  $k \in \mathbb{N}$  set*

$$A_{k,x} := \{y \in X : B_k(y) \text{ is isomorphic to } B_k(x)\}.$$

*Then for almost all  $x \in X$  one has*

$$\mu(A_{k,x}) > 0 \text{ for all } k \in \mathbb{N}.$$

*Proof.* For every  $k$  there are only finitely many distinct marked graphs appearing in the set  $\{B_{2(k+1)}(x) : x \in X\}$ . Let  $\mathcal{B}_k$  be the set of marked graphs  $B$  such that

$$\mu(\{x \in X : B_k(x) = B\}) > 0.$$

Set

$$M_k = \{x \in X : B_k(x) \in \mathcal{B}_k\}.$$

By construction,  $\mu(M_k) = 1$  and for every  $x \in M_k$  one has:  $\mu(A_{k,x}) > 0$ . Set

$$M = \bigcap_{k \in \mathbb{N}} M_k.$$

Then  $\mu(M) = 1$  and for every  $x \in M$  and every  $k \in \mathbb{N}$  one has:  $\mu(A_{k,x}) > 0$ , which finishes the proof.  $\square$

**Proof of Proposition 8.** By Corollary 2 it is sufficient to show that for almost all  $x \in X$  for all positive  $m \in \mathbb{C}[G]$  one has  $\|\rho_x(m)\| \leq \|\kappa(m)\|$ . Let  $\text{supp}(m) = \{g_1, \dots, g_n\}$  and  $A_{k,x}$  be the sets defined in Lemma 3. Till the end of the proof fix  $x$  such that

$$\mu(A_{k,x}) > 0 \text{ for all } k \in \mathbb{N}.$$

Let  $m \in \mathbb{C}[G]$  be positive. Without loss of generality we will assume that  $\|\rho_x(m)\| = 1$ . Let  $\epsilon > 0$ . Since

$$\sup\{(\rho_x(m)\xi, \xi) : \xi \in l^2(\Gamma_x), \|\xi\| = 1\} = 1$$

we can find a unit vector  $\eta \in l^2(\Gamma_x)$  of finite support such that  $(\rho_x(m)\eta, \eta) > 1 - \epsilon$ .

Further, fix  $k$  such that  $\text{supp}(\eta) \subset B_k(x)$ . Chose  $K$  such that  $gB_k(x) \subset B_K(x)$  for all  $g \in \text{supp}(m)$ . Observe that for every  $y \in A_{K,x}$ , any  $i, j \leq k$  and any  $g, h \in \text{supp}(m)$  one has:

$$gs_i y = hs_j y \Leftrightarrow gs_i x = hs_j x.$$

Using Lemma 2 successively for all elements of the form  $s_j^{-1}h^{-1}gs_i$ , where  $i, j \leq k$  and  $g, h \in \text{supp}(m)$  such that  $s_j^{-1}h^{-1}gs_i x \neq x$  we can find  $B \subset A_{K,x}$  such that  $\mu(B) > 0$  and

$$\mu(gs_i B \cap hs_j B) = 0 \text{ for all such } g, h, s_i, s_j.$$

Further, subdivide the set of positive numbers  $\mathbb{R}_+$  into subintervals

$$I_s = [(1 + \epsilon)^s, (1 + \epsilon)^{s+1}), s \in \mathbb{Z}$$

so that for every  $s$  one has  $ab^{-1} \in (1 - \epsilon, 1 + \epsilon)$  for  $a, b \in I_s$ . For every function  $f : \{1, \dots, k\} \times \text{supp}(m) \rightarrow \mathbb{Z}$  introduce the set

$$B_f = \{t \in B : \sqrt{\frac{d\mu(gs_i t)}{d\mu(s_i t)}} \in I_{f(i,g)} \text{ for every } g \in \text{supp}(m), 1 \leq i \leq k\}.$$

Since union of the sets  $B_f$  is the set  $B$  of positive measure for some  $f$  one has  $\mu(B_f) > 0$ . Fix such  $f$ . Then for every  $g \in \text{supp}(m)$ ,  $1 \leq i \leq k$  we have:

$$\sqrt{\frac{\mu(gs_i B_f)}{\mu(s_i B_f)}} \in I_{f(i,g)} \text{ and } \left| 1 - \sqrt{\frac{d\mu(t)}{d\mu(gt)}} \sqrt{\frac{\mu(gs_i B_f)}{\mu(s_i B_f)}} \right| < \epsilon \text{ for all } t \in s_i B_f.$$

Finally, for  $1 \leq i \leq k, g \in \text{supp}(m)$  and  $y = s_i x$  set  $e_y = \frac{1}{\sqrt{\mu(g s_i B_f)}} \mathbb{1}_{g s_i B_f}$ . Observe that

$$\|e_y - \kappa(s_i)e_x\|^2 = \int_{s_i B_f} \left( \frac{1}{\sqrt{\mu(s_i B_f)}} - \frac{1}{\sqrt{\mu(B_f)}} \sqrt{\frac{d\mu(s_i^{-1}t)}{d\mu(t)}} \right)^2 d\mu(t) < \epsilon. \quad (6)$$

Consider the spaces

$$\begin{aligned} \mathcal{H}_x &= \text{Span}\{\delta_y : y = g s_i x, i \leq k, g \in \text{supp}(m)\} \subset l^2(Gx), \\ \mathcal{L}_x &= \text{Span}\{e_y : y = g s_i x, i \leq k, g \in \text{supp}(m)\} \subset L^2(X, \mu). \end{aligned}$$

The map  $\phi : \delta_y \rightarrow e_y$  induces an isometry between these spaces. Moreover, by inequality (6) for every  $h \in \text{supp}(m)$  and every  $y = s_i x \in B_k(x)$ , where  $i \leq k$ , we have

$$\begin{aligned} \|\phi(\rho_x(h)\delta_y) - \kappa(h)e_y\|^2 &= \|e_{hy} - \kappa(h)e_y\|^2 \\ &\leq (\|e_{hs_ix} - \kappa(hs_i)e_x\| + \|\kappa(h)(e_{s_ix} - \kappa(s_i)e_x)\|)^2 \leq 4\epsilon. \end{aligned}$$

This implies that

$$\|\phi(\rho_x(h)\eta) - \kappa(h)\phi(\eta)\| \leq 2\sqrt{\epsilon}$$

for every  $h \in \text{supp}(m)$  and thus

$$\|\phi(\rho_x(m)\eta) - \kappa(m)\phi(\eta)\| \leq 2\sqrt{\epsilon}\|m\|_1, \quad (7)$$

where  $\|m\|_1 = \sum_{h \in \text{supp}(m)} |m(h)|$ . Since  $\|\rho_x(m)\eta\| \geq 1 - \epsilon$  when  $\epsilon \rightarrow 0$  we obtain that  $\|\kappa(m)\| = 1$  and  $1 \in \sigma(\kappa(m))$ . This finishes the proof of part 1) of Proposition 8 concerning the representation  $\kappa$ .

Now let  $\mu$  be  $G$ -invariant and non-atomic. From ergodicity one can deduce that the orbit  $Gy$  is infinite for almost all  $y \in X$ . Without loss of generality let  $Gx$  be infinite. Fix  $\eta \in l^2(\Gamma_x)$  be as above. Assume that

$$\alpha = \sum_{y \in \text{supp}(\eta)} \eta(y) \neq 0.$$

Then choose arbitrarily a sequence of distinct elements  $y_i$  from  $Gx \setminus \text{supp}(\eta)$  and for  $n \in \mathbb{N}$  introduce

$$\eta_n = \eta - \frac{\alpha}{n} \sum_{i=1}^n \delta_{y_i}, \quad m_n = m + \frac{1}{n^2} \sum_{i=1}^n \delta_{h_i},$$

where  $h_i \in G$  are such that  $h_i x = y_i$ . Clearly,

$$\sum_{y \in \text{supp}(\eta_n)} \eta_n(y) = 0, \quad \lim_{n \rightarrow \infty} \eta_n = \eta.$$



Moreover,  $\lim_{n \rightarrow \infty} \rho_x(m_n) = \rho_x(m)$  and  $\lim_{n \rightarrow \infty} \kappa(m_n) = \kappa(m)$  where the limits are in the strong operator topology. Therefore, without loss of generality we may assume that

$$\sum_{y \in \text{supp}(\eta)} \eta(y) = 0.$$

Then by construction  $\phi(\eta) \in L^2(X, \mu)$  is orthogonal to constant functions, and thus the representation  $\kappa$  in the inequality (7) can be replaced by  $\kappa_0$ . When  $\epsilon \rightarrow 0$  we obtain that  $\|\kappa_0(m)\| = 1$ . This finishes the proof of part 2) of Proposition 8.  $\square$

**Remark 2.** The condition of non-atomicity in the second part of Proposition 8 is necessary to avoid trivial counterexamples. Consider  $G = \mathbb{Z}^2 = \{0, 1\}$ . Equip  $X = \mathbb{Z}_2$  with the uniform probability measure  $\mu$ . Let  $G$  act on  $(X, \mu)$  by shifts. Then for any  $m = \alpha\delta_0 + \beta\delta_1 \in \mathbb{C}[G]$  one has:

$$\sigma(\rho_0(m)) = \sigma(\rho_1(m)) = \{1, \alpha - \beta\}, \quad \sigma(\kappa_0)(m) = \{\alpha - \beta\}$$

and thus the two spectra do not coincide when  $\alpha - \beta \neq 1$ .

### 3.3 Equivalence of Koopman and groupoid representations for a hyperfinite action.

**Proposition 9.** *For a hyperfinite measure class preserving action of  $G$  on a standard probability space  $(X, \mu)$  one has  $\kappa \sim \pi$ .*

In the proof we will use the following result (see [9]):

**Theorem 8.** *Let  $U$  be an aperiodic measure class preserving transformation of a Lebesgue space  $(X, \mu)$ . Then for any  $N$  and any  $\epsilon > 0$  there exists a measurable set  $A$  such that the sets  $A, UA, \dots, U^{N-1}A$  are pairwise disjoint and  $\mu(A \cup UA \cup \dots \cup U^{N-1}A) > 1 - \epsilon$ .*

Theorem 8 is a generalization of the famous Rohlin Lemma (see [46]) for the case of quasi-invariant measures.

**Proof of Proposition 9.** First notice that for the case of a finite  $X$  the groupoid representation  $\pi_X$  is unitary equivalent to a direct sum of finitely many copies of the Koopman representation  $\kappa_X$ . Therefore, without loss of generality we can assume that for all  $x \in X$  the orbit  $Gx$  is infinite. Since Koopman representation contains a Rodon-Nikodim derivative in the definition for us it will be convenient to replace  $\pi$  by a unitary equivalent representation  $\tilde{\pi}$  (see (3)). By Corollary 2 and part 1) of Theorem 1 it is sufficient to show that  $\|\kappa(m)\| \leq \|\tilde{\pi}(m)\|$  for every positive element  $m \in \mathbb{C}[G]$ .

Without loss of generality we will assume that  $\|\kappa(m)\| = 1$ . Since the action of  $G$  on  $X$  is hyperfinite there exists a measure-preserving automorphism  $U$  generating the equivalence relation  $\mathcal{R}$  generated by  $G$  on  $X$  (see e.g. [16], Proposition 4.1). Clearly,  $U$  is aperiodic. For  $g \in G$  and  $x \in X$  denote by  $n_g(x)$  the integer number such that  $gx = U^{n_g(x)}x$ . Observe that for every  $g \in G$  the function  $n_g(x)$  is measurable. Fix  $\delta > 0$ . Let  $\eta \in L^2(X, \mu)$  be a unit vector such that  $(\kappa(m)\eta, \eta) > 1 - \delta$ . Without loss of generality we may assume that the set of values of  $\eta$  is finite. Set  $K = \max\{\|\eta\|_\infty, 1\}$ .

Further, find a number  $L$  such that

$$\mu(\{x : |n_{g^{\pm 1}}(x)| \leq L \text{ for all } g \in \text{supp}(m)\}) \geq 1 - \frac{\delta}{2K^2}.$$

Introduce a set

$$\Omega = \{x : |n_{g^{\pm 1}}(x)| \leq L \text{ for all } g \in \text{supp}(m)\}.$$

Choose  $N$  such that  $\frac{L}{N} \leq \frac{\delta}{8K^2}$ . Using Theorem 8 one can construct a set  $C$  such that the sets  $C, UC, \dots, U^{N-1}C$  are pairwise disjoint, and

$$\mu(C \cup UC \cup \dots \cup U^{N-1}C) \geq 1 - \frac{\delta}{4K^2}.$$

Set  $C_j = U^j(C)$  for  $j = 0, 1, \dots, N-1$ . Let  $\Sigma = (C_L \cup C_{L+1} \cup \dots \cup C_{N-L-1}) \cap \Omega$ . Then  $\mu(\Sigma) \geq 1 - \frac{\delta}{K^2}$ . Consider the functions

$$\tilde{\eta}(x, y) = \eta(x) \mathbb{1}_C(y) \sum_{j=0}^{N-1} \delta_{x, U^j(y)},$$

where  $\delta_{x,y}$  is the Kroeneker delta symbol,

$$\tilde{\eta}_0(x, y) = \mathbb{1}_\Sigma(x) \tilde{\eta}(x, y) \text{ and } \eta_0(x) = \mathbb{1}_\Sigma(x) \eta(x),$$

where  $\mathbb{1}_A$  stands for the characteristic function of a set  $A$ . Observe that for every  $x \in X$  there exists at most one  $y$  such that  $\tilde{\eta}(x, y) \neq 0$ . By definition of  $\nu_l$  one has:

$$\|\tilde{\eta}\|^2 = \int_X \sum_{y \sim x} |\tilde{\eta}(x, y)|^2 d\mu(x) = \sum_{j=0}^{N-1} \int_{C_j} |\eta(x)|^2 d\mu(x) \leq \|\eta\|^2.$$

Similarly one can show that  $\|\tilde{\eta}_0\| = \|\eta_0\|$ . Since  $\mu(\Sigma) > 1 - \delta$  we obtain that

$$\|\eta\|^2 - \delta \leq \|\tilde{\eta}_0\|^2 \leq \|\tilde{\eta}\|^2 \leq \|\eta\|^2.$$

Let  $g \in \text{supp}(m)$ . Assume that  $x \in C_j \cap A$ , where  $L \leq j \leq N - L - 1$ . Let  $y = U^{-j}(x)$ . One has  $-L \leq n_{g^{-1}}(x) \leq L$  and  $g^{-1}x \in C_{j+n_{g^{-1}}(x)}$ . It follows that

$$\tilde{\eta}_0(x, y) = \eta_0(x), \quad \tilde{\eta}(g^{-1}x, y) = \eta(g^{-1}x).$$

If  $x \notin \Sigma$  then  $\eta_0(x) = 0$  and  $\tilde{\eta}_0(x, y) = 0$  for all  $y \sim x$ . We obtain:

$$\begin{aligned} (\tilde{\pi}(g)\tilde{\eta}, \tilde{\eta}_0) &= \int_X \sum_{y \sim x} \sqrt{\frac{d\mu(g^{-1}(x))}{d\mu(x)}} \tilde{\eta}(g^{-1}x, y) \overline{\tilde{\eta}_0(x, y)} d\mu(x) = \\ &= \int_X \sqrt{\frac{d\mu(g^{-1}(x))}{d\mu(x)}} \eta(g^{-1}x) \eta_0(x) d\mu(x) = (\kappa(g)\eta, \eta_0). \end{aligned}$$

Since  $\|\tilde{\eta} - \tilde{\eta}_0\| \leq \|\eta - \eta_0\| \leq \delta^{\frac{1}{2}}$  the latter implies that

$$|(\pi(g)\tilde{\eta}, \tilde{\eta}) - (\kappa(g)\eta, \eta)| \leq 2\delta^{\frac{1}{2}}.$$

Finally, we get:

$$|(\tilde{\pi}(m)\tilde{\eta}, \tilde{\eta}) - (\kappa(m)\eta, \eta)| \leq 2\delta^{\frac{1}{2}}\|m\|_1, \quad \text{where } \|m\|_1 = \sum_{g \in \text{supp}(m)} |m(g)|.$$

Since  $\delta > 0$  is arbitrary, the inequality  $\|\pi(m)\| \geq \|\kappa(m)\|$  follows. This finishes the proof.  $\square$

We finish this section by deriving Kuhn's Theorem 5 from Theorem 1.

**Proof of Theorem 5.** Let  $G$  acts ergodically by measure class preserving automorphisms on a probability measure space  $(X, \mu)$ . Assume that this action is Zimmer amenable. Then by Theorem 1 the corresponding Koopman representation is weakly equivalent to the quasi-regular representation  $\rho_x$  of  $G$  for almost every  $x$ . By result of Adams, Elliott and Giorgano [1] Zimmer's amenability implies that  $\text{St}_G(x)$  is amenable for almost every  $x$ . Let  $x$  be such that  $\rho_x \sim \kappa$  and  $\text{St}_G(x)$  is amenable. By Proposition 3,  $\rho_x \sim \lambda_G$ . This shows that  $\lambda_G \sim \kappa$ . In particular,  $\kappa \prec \lambda_G$ .  $\square$

## 4 Applications to weakly branch groups.

Let us first give a brief introduction to actions of groups on boundaries of regular rooted trees. We refer the reader to [25] for detailed definitions and properties of these actions.

A  $d$ -regular rooted tree is a tree  $T$ , with vertex set divided into levels  $V_n$ ,  $n \in \mathbb{Z}_+$ , such that  $V_0$  consists of one vertex  $v_0$  (called the root of  $T$ ), the edges are only between consecutive levels, and each vertex from  $V_n$ ,  $n \geq 0$  (we consider infinite trees), is connected by an edge to exactly  $d$  vertices from  $V_{n+1}$  (and one vertex from  $V_{n-1}$  for  $n \geq 1$ ). An automorphism of a rooted tree  $T$  is any automorphism of the graph  $T$  preserving the root. Denote by  $\text{Aut}(T)$  the group of automorphisms of  $T$ .

**Definition 4.** Let  $T$  be a  $d$ -regular rooted tree,  $d \geq 2$ , and  $G < \text{Aut}(T)$ . Rigid stabilizer of a vertex  $v$  is the subgroup  $\text{rist}_v(G) = \{g \in G : \text{supp}(g) \subset T_v\}$ . Rigid stabilizer of level  $n$  is

$$\text{rist}_n(G) = \prod_{v \in V_n} \text{rist}_v(G).$$

$G$  is called *branch* if it is transitive on each level and  $\text{rist}_n(G)$  is a subgroup of finite index in  $G$  for all  $n$ .  $G$  is called *weakly branch* if it is transitive on each level  $V_n$  of  $T$  and  $\text{rist}_v(G)$  is nontrivial for each  $v$ .

For each level  $V_n$  of a  $d$ -regular rooted tree an automorphism  $g$  of  $T$  can be presented in the form

$$g = \sigma \cdot (g_1, \dots, g_{d^n}), \quad (8)$$

where  $\sigma \in \text{Sym}(V_n)$  is a permutation of the vertices from  $V_n$  and  $g_i$  are the restrictions of  $g$  on the subtrees emerging from the vertices of  $V_n$ .

**Definition 5.** For an element  $g \in \text{Aut}(T)$  denote by  $k_n(g)$  the number of restrictions  $g_i$  to the vertices of level  $n$  not equal to identity automorphism. We will call  $g$  subexponentially bounded if for every  $0 < \gamma < 1$  one has

$$\lim_{n \rightarrow \infty} k_n(g) \gamma^n = 0.$$

A group  $G < \text{Aut}(T)$  is subexponentially bounded if each  $g \in G$  is subexponentially bounded.

For a  $d$ -regular rooted tree  $T$  its boundary  $\partial T$  is the set of infinite paths starting from  $v_0$ . Observe that  $\partial T$  can be identified with a space of sequences  $\{x_j\}_{j \in \mathbb{N}}$  where  $x_j \in \{1, \dots, d\}$ . For a vertex  $v$  of  $T$  we will denote by  $\partial T_v \subset \partial T$  the set of paths passing through  $v$ . Supply  $\partial T$  by the topology generated by the sets  $\partial T_v$ . Automorphisms of  $T$  act naturally on  $\partial T$  by homeomorphisms. Notice that  $\partial T$  admits a unique  $\text{Aut}(T)$ -invariant measure  $\mu$ . This measure is uniform in the sense that

$$\mu(\partial T_v) = \frac{1}{d^n} \text{ for any } n \text{ and any } v \in V_n.$$

Grigorchuk, Nekrashevich and Suschanski showed (under more general settings) that this measure is ergodic with respect to a group  $G < \text{Aut}(T)$  if and only if the action of  $G$  is transitive on each level  $V_n$  of  $T$ . Moreover, in this case it is uniquely ergodic. We refer the reader to [29], Proposition 6.5 for details.

Further, let  $G$  be a weakly branch group acting on  $d$ -regular rooted tree  $T$ . Denote by  $\mathcal{O}$  the set of orbits of  $G$  on  $X$ . For  $\sigma \in \partial T$  denote by  $\rho_\sigma$  the corresponding quasi-regular representation of  $G$ . Let

$$\mathcal{P} = \{p = (p_1, p_2, \dots, p_d) : p_i > 0 \text{ for } i = 1, 2, \dots, d \text{ and } \sum_{i=1}^d p_i = 1\} \quad (9)$$

be the set of all probability distributions on the alphabet  $\{1, 2, \dots, d\}$  assigning positive probability to every letter and

$$\mathcal{P}^* = \{p \in \mathcal{P} : p_i \neq p_j \text{ for all } 1 \leq i < j \leq d\}. \quad (10)$$

For  $p \in \mathcal{P}^*$  denote by  $\mu_p = \prod_{\mathbb{N}} p$  the corresponding Bernoulli measure on  $\partial T$  and by  $\kappa_p$  the Koopman representation associated to the action of  $G$  on  $(\partial T, \mu_p)$ .

Using Mackey criterion of irreducibility of quasi-regular representations Bartholdi and Grigorchuk in [4] showed that quasi-regular representations  $\rho_\sigma$  corresponding to the action of a weakly branch group  $G$  on the boundary of a rooted tree are irreducible. Moreover, in [14] the authors proved the following.

**Theorem 9.** *Let  $G$  be a subexponentially bounded weakly branch group acting on the boundary of a  $d$ -regular rooted tree  $T$ . For every  $p \in \mathcal{P}^*$  the representation  $\kappa_p$  of is irreducible. Moreover, the representations of  $G$  from the set  $\{\kappa_p : p \in \mathcal{P}^*\} \cup \{\rho_\sigma : \sigma \in \mathcal{O}\}$  are pairwise disjoint.*

For  $g \in \text{Aut}(T)$  a point  $x = x_1 x_2 x_3 \dots \in \partial T$  is called  $g$ -rigid if there exists  $n \in \mathbb{N}$  and  $v = x_1 x_2 \dots x_n \in V_n$  such that the restriction  $g|_{\partial T_v}$  is trivial. For  $G < \text{Aut}(T)$  denote by  $R(G)$  the set of points  $x \in \partial T$  such that  $x$  is  $g$ -rigid for all  $g \in G$ . Such points are called rigid. Let  $\mathcal{O}_{R(G)}$  be the set of  $G$ -orbits of points from  $R(G)$ . From the proof of Proposition 2 of [14] we obtain:

**Lemma 4.** *Let  $T$  be a  $d$ -regular rooted tree and  $G < \text{Aut}(T)$  be subexponentially bounded. Then for any  $p \in \mathcal{P}$  (see (9)) one has  $\mu_{\mathcal{P}}(R(G)) = 1$ .*

Recall that for an action of a group  $G$  by homeomorphisms on a topological space  $X$  a point  $x$  is called *typical* if for every  $g \in G$  either  $gx \neq x$  or  $g$  acts trivially on some neighborhood of  $x$ . Clearly, the set of all typical points is open and  $G$ -invariant. Observe that for  $G < \text{Aut}(T)$  where  $T$  is a regular rooted tree any rigid point  $x \in \partial T$  is typical. The next proposition is a topological version of Proposition 5 and is a generalization of Proposition 6.21 from [29] (see also [25], Proposition 8.8).

**Proposition 10.** *Let  $G$  act minimally on a topological space  $X$ . Let  $n \in \mathbb{N}$  and  $g_1, \dots, g_n \in G$ . Then for any typical points  $x, y \in X$  the orbital graphs  $\Gamma_{x, g_1, \dots, g_n}$  and  $\Gamma_{y, g_1, \dots, g_n}$  are locally isomorphic.*

*Proof.* Let  $A$  be the set of all typical points. Fix  $r \in \mathbb{N}$ . Denote by  $D_r$  the set of finite rooted marked graphs  $\Delta$  such that there exists  $x \in A$  for which  $B_r(x) = \Delta$ . For any  $\Delta \in D_r$  denote by  $X_\Delta(r)$  the set of  $x \in A$  such that  $B_r(x) = \Delta$ . Given a point  $x \in A$ ,  $1 \leq l, m \leq r$  and  $1 \leq i \leq n$  by definition of a typical point there exists a neighborhood  $U(x)$  such that either

- a) for all  $y \in U(x)$  one has  $s_m^{-1}g_i s_l y = y$  (i.e. vertices  $s_l y$  and  $s_m y$  are connected by an edge marked by  $g_i$  in  $\Gamma_y$ ) or  
b) for all  $y \in U(x)$  one has  $s_m^{-1}g_i s_l y \neq y$  (i.e. vertices  $s_l y$  and  $s_m y$  are not connected by an edge marked by  $g_i$  in  $\Gamma_y$ ).

It follows that the sets  $X_\Delta(r)$  are open.

Further, let  $x, y \in A$  and  $\Delta = B_r(x)$ . By minimality of the action of  $G$  on  $X$  there exists  $g \in G$  such that  $gy \in X_\Delta$ . Thus,  $B_r(x) = \Delta = B_r(gy)$  which finishes the proof.  $\square$

Combining these results with Theorem 1 and taking into account that any subexponentially bounded weakly branch group generates a hyperfinite equivalence relation on  $\partial T$  (see [27], Theorem of Section 3) we obtain:

**Corollary 3.** *For any subexponentially bounded weakly branch group  $G$  acting on a  $d$ -regular rooted tree the representations from the set  $\{\kappa_p : p \in \mathcal{P}^*\} \cup \{\rho_\sigma : \sigma \in \mathcal{O}_{R(G)}\}$  are irreducible pairwise disjoint (not unitary equivalent) and pairwise weakly equivalent.*

Observe that the sets  $\mathcal{P}^*$  and  $R(G)$  have cardinality of continuum. Similar results are known for free groups  $F_n$ ,  $n \geq 2$ . Namely, for every  $n \geq 2$  there exists a continuum of irreducible pairwise disjoint and pairwise weakly equivalent Koopman type representations of  $F_n$  (see e.g. [39]). However, weak equivalence of these representations uses the fact that the reduced  $C^*$ -algebra of  $F_n$  is simple (see [44]). The class of weakly branch groups contains many amenable groups. Recall that groups of intermediate growth are amenable but not elementary amenable. For any amenable group the reduced  $C^*$ -algebra is not simple (see e.g. [31]). To the best authors' knowledge Corollary 3 gives the first example of amenable groups admitting a continuum of pairwise disjoint weakly equivalent irreducible representations. Notice that all  $p$ -groups constructed in [22] and [23] (for each prime  $p$  there is  $2^{x_0}$  such groups) are bounded (and therefore subexponentially bounded) branch groups and hence satisfy the conditions of Corollary 3.

## 5 Examples and proof of Theorem 2.

One of basic examples of branch groups is the group  $\mathcal{G}$  constructed by the second author [21] and studied in [22] and many other papers. This group acts on the boundary of the binary (2-regular) rooted tree (which we will denote by  $T$  in this subsection) and is generated by elements  $a, b, c, d$  satisfying the following recursions:

$$a = \sigma \cdot (\text{Id}, \text{Id}), \quad b = (a, c), \quad c = (a, d), \quad d = (\text{Id}, b), \quad (11)$$

where  $\text{Id}$  is the identical action. The group  $\mathcal{G}$  satisfies many unusual properties and answers a number of open questions. For example,  $\mathcal{G}$  is an infinite finitely generated torsion group (answer to one of the Burnside problems whether a finitely generated group in which every element has finite order must necessarily be a finite group), is the first example of intermediate growth groups, (answer to Milnor's problem) and is amenable but not elementary amenable (answer to the question posed by Mahlon Day's). We refer the reader to [22] and [42] for details.

Set

$$\Delta = \frac{1}{4}(a + b + c + d) \in \mathbb{C}[\mathcal{G}]. \quad (12)$$

Let  $\kappa$  be the Koopman representation corresponding to the action of  $\mathcal{G}$  on  $(\partial T, \mu)$ , where  $\mu$  is the unique probability invariant measure on  $\partial T$ . In [4] Bartholdi and the second author developed a method for calculating spectra of Hecke type operators and showed the following:

**Theorem 10.** *For all  $x \in \partial T$  one has  $\sigma(\kappa(\Delta)) = \sigma(\rho_x(\Delta)) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1] \subset \sigma(\lambda_{\mathcal{G}}(\Delta))$ .*

Also they calculated spectra of similar operators for other groups and constructed examples of groups acting on rooted trees with interesting spectral properties. The main tools they used were operator recursions, Schur complement and the reduction of the spectral problem to the problem of finding a suitable invariant set for the associated rational map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  for some  $n$ .

Notice that the spectrum of  $\rho_x(\Delta)$  coincides with the spectrum of the Schreier graph  $\Gamma_x$  for every  $x \in \partial T$ . An important question related to the Schreier graph  $\Gamma_x$ ,  $x \in \partial T$ , is computing the associated Kesten measure, i.e. the spectral measure of  $\rho_x(\Delta)$  associated to the unit vector  $\delta_x \in l^2(Gx)$ . Kesten measures for the action of  $\mathcal{G}$  on  $\partial T$  were computed in [30].

Here we will compute the spectrum of the Cayley graph of  $\mathcal{G}$ . Also, using operator recursions similar to those from [4] we will prove the results analogous to Theorem 10 for the Koopman representations of  $\mathcal{G}$  corresponding to quasi-invariant Bernoulli measures on  $\partial T$  and for the groupoid representation of  $\mathcal{G}$  corresponding to the invariant Bernoulli measure on  $\partial T$ . Surprisingly, the spectrum does not depend on the parameter  $p$ ,  $0 < p < 1$ , defining the measure.

## 5.1 Spectrum of $\mathcal{G}$ .

Here we prove Theorem 2 which is equivalent to:

$$\sigma(\lambda_{\mathcal{G}}(\Delta)) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1],$$

where  $\lambda_{\mathcal{G}}$  is the regular representation of  $\mathcal{G}$ .

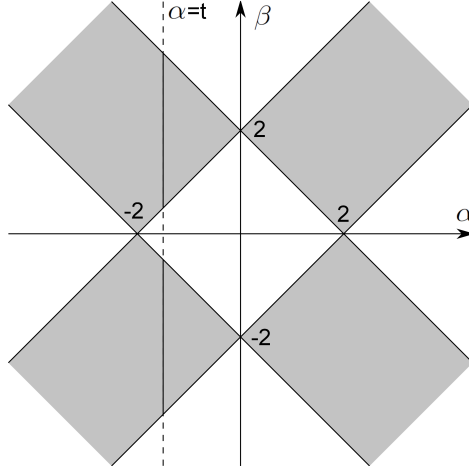


Figure 1: The set  $\Omega$ .

Introduce a 2-parameter family of elements  $q(\alpha, \beta) = 4\Delta - (\alpha + 1)a - (\beta + 1)e = -\alpha a + b + c + d - (\beta + 1)e \in \mathbb{C}[\mathcal{G}]$ , where  $e$  is the unit of  $\mathcal{G}$ . For a unitary representation  $\rho$  of  $\mathcal{G}$  let  $\Sigma_\rho$  be the set of pairs  $(\alpha, \beta) \in \mathbb{R}^2$  such that  $\rho(q(\alpha, \beta))$  is not invertible.

**Lemma 5.** *For any unitary representation  $\rho$  one has:  $\Sigma_\rho \subset \Omega = \{(\alpha, \beta) : ||\alpha| - |\beta|| \leq 2, ||\alpha| + |\beta|| \geq 2\}$ .*

*Proof.* It is straightforward to verify that  $(b + c + d - e)^2 = 4e \in \mathbb{C}[\mathcal{G}]$ , where  $e \in \mathcal{G}$  is the group unit. Let  $A = \rho(a)$ ,  $U = \rho(\frac{1}{2}(b + c + d - e))$ . Then  $A$  and  $U$  are unitary operators such that  $A^2 = U^2 = \text{Id}$ . For any  $\alpha, \beta \in \mathbb{R}$  one has:

$$\rho(q(\alpha, \beta)) = -\alpha A + 2U - \beta \text{Id}.$$

If  $|\alpha| + |\beta| < 2$  then

$$-\alpha A + 2U - \beta \text{Id} = U(2\text{Id} - \alpha U A - \beta U)$$

is invertible since  $\|\alpha U A + \beta U\| \leq |\alpha| + |\beta| < 2$ . If  $|\alpha| > |\beta| + 2$  then

$$-\alpha A + 2U - \beta \text{Id} = A(-\alpha \text{Id} + 2AU - \beta A)$$

is invertible since  $\|2AU - \beta A\| \leq 2 + |\beta| < |\alpha|$ . Finally, if  $|\beta| > |\alpha| + 2$  then  $-\alpha A + 2U - \beta \text{Id}$  is invertible since  $\|-\alpha A + 2U\| \leq |\alpha| + 2 < |\beta|$ .  $\square$

*Proof of Theorem 2.* By construction, spectrum of  $\lambda_{\mathcal{G}}(4\Delta - e)$  coincides with the intersection of  $\Sigma_{\lambda_{\mathcal{G}}}$  and the line  $\alpha = -1$ . By Lemma 5 we obtain  $\sigma(\lambda_{\mathcal{G}}(4\Delta - e)) \subset [-3, -1] \cup [1, 3]$ . It follows that  $\sigma(\lambda_{\mathcal{G}}(\Delta)) \subset [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$ . The opposite inclusion is contained in Theorem 10.  $\square$



In fact, calculations in [4] show that for any  $t \in \mathbb{R}$  one has

$$\sigma(\rho_x(-ta + b + c + d)) = \Lambda_t := (\{\alpha = t\} \cap \Omega) + 1$$

(for instance,  $\Lambda_t = [t - 1, -t - 1] \cup [t + 3, -t + 3]$  if  $-2 < t < 0$ ) which is a union of two intervals, an interval, or two points (if  $t = 0$ ). Arguments similar to the proof of Theorem 2 show that  $\sigma(\lambda_{\mathcal{G}}(-ta + b + c + d)) = \Lambda_t$  for any  $t \in \mathbb{R}$ .

## 5.2 Spectra of Koopman representations of $\mathcal{G}$ .

The boundary  $\partial T$  of a binary rooted tree is homeomorphic to a space of infinite sequences  $\{0, 1\}^{\mathbb{N}}$  and hence is homeomorphic to a Cantor set. For any  $q \in (0, 1)$  define a measure  $\nu_q$  on  $\{0, 1\}$  by

$$\nu_q(\{0\}) = q, \nu_q(\{1\}) = 1 - q.$$

Let  $\mu_q = \nu_q^{\mathbb{N}}$  be the corresponding Bernoulli measure on  $\partial T$ .

**Definition 6.** For a  $d$ -regular rooted tree  $T_d$  a subgroup  $G < \text{Aut}(T_d)$  is called self-similar if for every  $g \in G$  the restrictions  $g_1, g_2, \dots, g_d$  of  $g$  on the subtrees emerging from vertices of the first level also belong to  $G$  (after identification of the subtrees with the original tree).

The group  $\mathcal{G}$  is an example of self-similar groups (because of relations (11)). For  $q \in (0, 1)$  let  $\kappa_q$  be the Koopman representation associated to the action of  $\mathcal{G}$  on  $(\partial T, \mu_q)$ . We prove that the spectrum of  $\kappa_q(\Delta)$  (see (12)) does not depend on the parameter  $q$  and thus coincides with the spectrum given by Theorem 10. Observe that the representations  $\kappa_q$  for  $q \neq \frac{1}{2}$  are irreducible (see [14]), but  $\kappa_{\frac{1}{2}}$  is a direct sum of countably many finite-dimensional irreducible representations (see [4]).

**Proposition 11.** *For every  $q \in (0, 1)$  one has  $\sigma(\kappa_q(\Delta)) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$ .*

Fix  $q \in (0, 1), q \neq \frac{1}{2}$ . Set

$$A = \kappa_q(a), B = \kappa_q(b), C = \kappa_q(c), D = \kappa_q(d).$$

Consider the operators

$$Q(\alpha, \beta) = \kappa_q(4\Delta - (\alpha + 1)a - (\beta + 1)) = -\alpha A + B + C + D - (\beta + 1)\text{Id} \quad (13)$$

on  $L^2(\partial T, \mu_q)$ . Denote by  $\Sigma$  the set of pairs  $(\alpha, \beta) \in \mathbb{R}^2$  such that  $Q(\alpha, \beta)$  is not invertible. Following [4] we will prove a statement more general than Proposition 11:

**Proposition 12.**  $\Sigma = \{(\alpha, \beta) : ||\alpha| - |\beta|| \leq 2, ||\alpha| + |\beta|| \geq 2\}$

In [4] the authors considered restrictions  $Q_n(\alpha, \beta)$  of  $Q(\alpha, \beta)$  on  $\mathcal{G}$ -invariant finite dimensional subspaces and constructed operator recursions for  $Q_n(\alpha, \beta)$  to describes spectra of  $Q_n(\alpha, \beta)$  and  $Q(\alpha, \beta)$ . In the case of a quasi-invariant measure the representation  $\kappa_q$  is irreducible for  $q \neq \frac{1}{2}$  and so does not have invariant subspaces. We need to modify arguments from [4] and use operator recursions for infinite-dimensional Hilbert spaces. Recall that  $V_n$  is the set of vertices of level  $n$  in  $T$ . For every  $n$  encode vertices of  $V_n$  by  $\{0, 1\}^n$  so that for every vertex  $v = x_1 x_2 \dots x_n \in V_n$  one has:

$$\mu_q(\partial T_j) = q^{1-\sum x_i} (1-q)^{\sum x_i}.$$

Let  $v_0$  and  $v_1$  be the vertices of the first level of  $T$ . Denote

$$\mathcal{H} = L^2(\partial T, \mu_q), \quad \mathcal{H}_j = \{f \in \mathcal{H} : \text{supp}(f) \subset \partial T_{v_j}\}, \quad \text{where } j = 0, 1.$$

Observe that  $\mathcal{H}_0$  and  $\mathcal{H}_1$  are isomorphic to  $\mathcal{H}$  via the isometries  $I_j : \mathcal{H}_j \rightarrow \mathcal{H}, j = 0, 1$ , given by:

$$(I_0 f)(x) = \sqrt{q} f(0x), \quad (I_1 f)(x) = \sqrt{1-q} f(1x),$$

where  $x \in \partial T$  is encoded by sequences from  $\{0, 1\}^\infty$ . Using the decomposition  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  and identifying with  $\mathcal{H}$  the spaces  $\mathcal{H}_i$  using the isometries  $I_i, i = 0, 1$ , we can write every operator on  $\mathcal{H}$  in a  $2 \times 2$  block matrix form whose entries are also operators on  $\mathcal{H}$ . The operators of the Koopman representation  $\kappa_q$  corresponding to the generators of  $\mathcal{G}$  can be written as follows:

$$\begin{aligned} A &= \begin{bmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \\ C &= \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad D = \begin{bmatrix} \text{Id} & 0 \\ 0 & B \end{bmatrix}. \end{aligned} \tag{14}$$

In particular, the recursions do not depend on parameter  $q$ . It follows that the operator  $Q(\alpha, \beta)$  can be written as follows:

$$Q(\alpha, \beta) = \begin{bmatrix} 2A - \beta \text{Id} & -\alpha \text{Id} \\ -\alpha \text{Id} & B + C + D - (\beta + 1) \text{Id} \end{bmatrix}.$$

Notice that  $(2A - \beta \text{Id})(2A + \beta \text{Id}) = (4 - \beta^2) \text{Id}$ . Assume that  $\beta \neq \pm 2$ . Straightforward calculations show that

$$Q(\alpha, \beta) \begin{bmatrix} \text{Id} & \frac{\alpha(2A + \beta \text{Id})}{4 - \beta^2} \\ 0 & \text{Id} \end{bmatrix} = \begin{bmatrix} 2A - \beta \text{Id} & 0 \\ -\alpha \text{Id} & Q(\frac{2\alpha^2}{4 - \beta^2}, \beta + \frac{\alpha^2 \beta}{4 - \beta^2}) \end{bmatrix}. \tag{15}$$

Introduce a map on  $\mathbb{R}^2 \setminus \mathbb{R} \times \{\pm 2\}$  by

$$F(\alpha, \beta) = (\frac{2\alpha^2}{4 - \beta^2}, \beta + \frac{\alpha^2 \beta}{4 - \beta^2}).$$

Also, for  $n \in \mathbb{N}$  set  $(\alpha_n, \beta_n) = F^n(\alpha, \beta)$ . Since  $\sigma(A) = \{-1, 1\}$  we obtain

**Proposition 13.** *If  $\beta \neq \pm 2$  then  $(\alpha, \beta) \in \Sigma$  if and only if  $F(\alpha, \beta) \in \Sigma$ .*

Next, let us prove another auxiliary statement.

**Lemma 6.**  $\Sigma \supset \{(\alpha, \beta) : |\alpha - 2| = |\beta|\}$ .

*Proof.* Since  $(B + C + D - \text{Id})^2 = 4\text{Id}$  and clearly  $B + C + D - \text{Id}$  is not scalar we obtain that  $\sigma(B + C + D - \text{Id}) = \{-2, 2\}$ . Thus,  $-1$  and  $3$  are eigenvalues of the operator  $B + C + D$ . Let  $\eta \neq 0, \eta \in \mathcal{H}$  be such that  $(B + C + D - 3)\eta = 0$ . Let  $\alpha = \beta + 2$ . Then  $\alpha_1 = \beta_1 + 2$  and  $\beta_1 = \frac{4\beta}{2-\beta}$ . It is not hard to see that the map

$$h(z) = \frac{4z}{2-z}$$

has two fixed points on the Riemann sphere: a repelling fixed point  $0$  and an attracting fixed point  $-2$ , and for every  $z \neq 0$   $h^n(z) \rightarrow -2$  exponentially fast. Thus, if  $\beta \neq 0$ , then

$$\beta_n \rightarrow -2, \quad \alpha_n \rightarrow 0$$

exponentially fast. It follows that  $Q(\alpha_n, \beta_n)\eta \rightarrow 0$  exponentially fast.

Further, let  $\xi \in \mathcal{H}$ . Applying the operator in (15) to the block vector  $\begin{bmatrix} 0 \\ \xi \end{bmatrix}$ , where  $\xi \in \mathcal{H}$ , we obtain:

$$Q(\alpha, \beta) \begin{bmatrix} \frac{\alpha(2A+\beta\text{Id})}{4-\beta^2}\xi \\ \xi \end{bmatrix} = \begin{bmatrix} 0 \\ Q(\alpha_1, \beta_1)\xi \end{bmatrix}.$$

Thus, there exists  $\xi_1$  such that  $\|\xi_1\| \geq \|\xi\|$  and  $\|Q(\alpha_1, \beta_1)\xi\| = \|Q(\alpha, \beta)\xi_1\|$ . By induction, we get that  $\|Q(\alpha, \beta)\eta_n\| = \|Q(\alpha_n, \beta_n)\eta\|$  for some  $\eta_n$  with  $\|\eta_n\| \geq \|\eta\|$ . Since  $\|Q(\alpha_n, \beta_n)\eta\|$  converges to  $0$ , we obtain that  $Q(\alpha, \beta)$  is not invertible. The case  $\alpha = 2 - \beta$  can be treated similarly.  $\square$

*Proof of Proposition 11.* Following ideas from [4] consider the curves

$$\gamma_{n,j} = \{(\alpha, \beta) : 4 - \beta^2 + \alpha^2 - 4\alpha \cos(\frac{2\pi j}{2^n}) = 0\}.$$

Observe that  $\gamma_{0,j} = \{(\alpha, \beta) : |\alpha - 2| = |\beta|\}$ . Straightforward computations show that  $F(\gamma_{n,j}) \subset \gamma_{n-1,j}$  for all  $n, j \in \mathbb{N}$ . From Proposition 13 and Lemma 6 taking into account that  $\Sigma$  is closed we obtain that  $\Sigma \supset \gamma_{n,j}$  for all  $n, j$ . Notice that the curve  $\gamma_{n,j}$  can be written as:

$$\beta = \pm \sqrt{\alpha^2 - 4\alpha \cos(\frac{2\pi j}{2^n}) + 4}.$$

Since the union of curves  $\gamma_{n,j}$  is dense in the region

$$S = \{(\alpha, \beta) : ||\alpha| - |\beta|| \leq 2, ||\alpha| + |\beta|| \geq 2\}$$

we obtain that  $\Sigma \supset S$ . From Lemma 5 we deduce that  $\Sigma = S$ , which finishes the proof.  $\square$

### 5.3 Spectra of groupoid representations of $\mathcal{G}$ .

Let  $\pi$  be the groupoid representation of  $\mathcal{G}$  corresponding to the action of  $\mathcal{G}$  on  $(\partial T, \mu)$ , where  $\mu$  is the invariant Bernoulli measure on  $\partial T$ . The following Proposition follows from Theorem 10 and Theorem 1.

**Proposition 14.**  $\sigma(\pi(\Delta)) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$ .

To give another illustration of the method of operator recursions we provide a sketch of a direct proof of Proposition 14.

*Proof.* Let  $v_0$  and  $v_1$  be the vertices of the first level of  $T$ . For  $i, j \in \{0, 1\}$  introduce a subspace

$$\mathcal{H}_{i,j} = \{\eta \in L^2(\mathcal{R}, \nu) : \text{supp}(\eta) \subset \partial T_{v_i} \times \partial T_{v_j}\}.$$

One has:

$$L^2(\mathcal{R}, \nu) = \mathcal{H}_{0,0} \oplus \mathcal{H}_{1,0} \oplus \mathcal{H}_{0,1} \oplus \mathcal{H}_{1,1}.$$

Recall that  $x, y \in \partial T$  belong to the same orbit by  $\mathcal{G}$  if and only if  $x_i = y_i$  for all large enough  $i$  (see [25], Theorem 7.3). This implies that the subspaces  $\mathcal{H}_{i,j}$  are canonically isomorphic to  $L^2(\mathcal{R}, \nu)$ . Thus, every operator acting on  $L^2(\mathcal{R}, \nu)$  can be written in a  $4 \times 4$  block matrix form with entries operators on  $L^2(\mathcal{R}, \nu)$ . Set

$$A = \pi(a), B = \pi(b), C = \pi(c), D = \pi(d).$$

It is not hard to see that every operator  $X$  from the latter list can be written as

$$X = \begin{bmatrix} Y & 0_2 \\ 0_2 & Y \end{bmatrix},$$

where  $0_2$  is the  $2 \times 2$  zero matrix and  $Y$  is the  $2 \times 2$  block matrix representation for the corresponding operator from (14). Similarly to (13) introduce an operator

$$Q(\alpha, \beta) = \pi(4\Delta - (\alpha + 1)a - (\beta + 1)) = -\alpha A + B + C + D - (\beta + 1)\text{Id}$$

on  $L^2(\mathcal{R}, \nu)$  and denote by  $\Sigma$  the set of pairs  $(\alpha, \beta) \in \mathbb{R}^2$  such that  $Q(\alpha, \beta)$  is not invertible. One has:

$$Q(\alpha, \beta) = \begin{bmatrix} Y & 0_2 \\ 0_2 & Y \end{bmatrix}, \text{ where } Y = \begin{bmatrix} 2A - \beta \text{Id} & -\alpha \text{Id} \\ -\alpha \text{Id} & B + C + D - (\beta + 1)\text{Id} \end{bmatrix}.$$

Similarly to Proposition 12 one can show that  $\Sigma = \{(\alpha, \beta) : ||\alpha| - |\beta|| \leq 2, ||\alpha| + |\beta|| \geq 2\}$  from which the statement of Proposition 14 follows easily.  $\square$

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